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SIMPLE MODELS FOR POSITIVE-VALUED
AND DISCRETE-VALUED TIME SERIES
WITH ARMA CORRELATION STRUCTURE

by

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20. Abstract continued

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SIMPLE MODELS FOR POSITIVE-VALUED AND DISCRETE-VALUED TIME SERIES

WITH ARMA CORRELATION STRUCTURE

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Simple Models for Positive-Valued and Discrete-Valued Time Series with ARMA Correlation Structure

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Abstract

Three models for positive-valued and discrete-valued stationary time series are discussed. All have the property that for a range of specified marginal distributions the time series have the same correlation structure as the usual linear, autoregressive-moving average (ARMA) model. The models differ in the range of marginal distributions which can be accommodated and in the simplicity and flexibility of each model. Specifically the EARMA-type processes can be extended from the exponential distribution to a rather narrow range of continuous distributions; the DARMA-type processes can be defined usefully for any discrete marginal distribution and are simple and flexible. Finally the marginally controlled semi-Markov generated process can be defined for any continuous or discrete positive-valued distribution and is therefore very flexible. However the model suffers from some complexity and parametric obscurity.

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1. Introduction

In much of the current work on the analysis of stationary time series there is an implicit assumption that the marginal distribution of the time series is normal. The assumption is implicit in that the marginal distribution is not considered to be of interest per se in the analysis, and also in that the statistical procedures which are used are very definitely based on normality assumptions. The stationary model on which much of this time series analysis is based is the mixed autoregressive moving average process,

$$a_0X_i + a_1X_{i-1} + \dots + a_pX_{i-p} = b_0\epsilon_i + b_1\epsilon_i + \dots + b_q\epsilon_{i-q} \qquad i=0, \pm 1, \pm 2, \dots , \tag{1.1}$$

sometimes called the ARMA(p,q) or Box-Jenkins process. The process (1.1) is specified quite generally as a linear combination of i.i.d. random variables $\{\epsilon_i\}$ of unspecified distribution, the linear, additive structure determining the correlation structure of the stationary sequence $\{X_i\}$ under well-known restrictions on the parameters. If one wants $\{X_i\}$ to be a time series with normally distributed marginal distribution, this can be accomplished by taking the ϵ_i's to be normally distributed. The model is then completely specified.

There are, however, many situations in which observations occur serially and in which the marginal distribution is patently non-normal. For example, data on the number of occurrences of all known diseases in each week is kept by the National Center for Health Statistics. The data is not only discrete count data, but for many diseases it is mostly on the order of 0, 1, 2, 3, and very seldom above this.

It has been suggested that such non-normal data be handled by data transformations and this is probably appropriate if the data is only slightly non-normal. In other cases it seems reasonable to start afresh and develop models from scratch. In this paper we summarize attempts to do this for stationary time series which are known to be non-normal because of either positivity or discreteness or both. The essence of the models is that the marginal distribution is specified, as well as the correlation structure. More generally the models are required to be simple and flexible in the following senses:

a) The models should be specified in terms of easily observed and measured quantifiers. When the models are stationary, these quantifiers would typically be

   i) the marginal distribution, and

   ii) second-order moments (correlations).

b) The models should be parametrically parsimonious and hopefully parametrically meaningful.

c) The models should be easy to generate on computers, i.e., they should be structurally simple; in fact it might be preferable for the models to have linear structure.

d) The models should be easy to fit to data, both informally and formally.

The model (1.1) certainly has most of the above features, but it is not known in general how to specify the distribution of ϵ_i so as to produce a given, continuous marginal distribution for the X_i's. Moreover, it is clearly not possible to do this at all if the X_i's are discrete random variables.

The work described in this paper on non-normal time series is joint work with D. P. Gaver, P. A. Jacobs and A. J. Lawrance. Although the work has much broader connotation, it will be described in the context in which it arose, that of the description of stochastic point processes, or series of events occurring in time. One way in which these point processes can be described is as a sequence of intervals between events $\{X_i\}$, which are of course positive-valued random variables. In the common case of a Poisson point process the X_i's have an exponential distribution. However, as in the case of epidemics, point processes are generally observed as counts of events in successive fixed intervals and these are non-negative discrete valued random variables. For the Poisson process these counts are independent and Poisson distributed and this serves as the null model in the analysis of count data from point processes.

Three distinct models are discussed in the context of the analysis and description of point processes. All of them satisfy the requirements discussed above to some degree. The EARMA-type process described first has recently been extended to have a complete ARMA-type correlation structure, but the process cannot be extended to all continuous marginal distributions. Marginally controlled semi-Markov generated processes, on the other hand, give a complete analog to (1.1), but they do not have linear structure. They can also be extended to give processes with discrete marginal distributions. A simpler, random linear structure has been derived, however, which gives discrete processes with ARMA structure. These are DARMA-type processes and come closer than the other processes to fulfilling the requirements of simplicity and flexibility.

Further details on the processes are to be found in the references.

2. Interval Models: Sequences of continuous positive-valued random variables

Univariate point processes in continuous time can be described equally well through the structure of the intervals between events $\{X_i\}$, where the X_i's are continuous and positive-valued random variables, or the counting process $\{N(t)\}$, where $N(t)$ gives the number of events in $(0,t]$ and is discrete and non-negative. We discuss the modelling of the intervals $\{X_i\}$ first. Of course the applications of the models are much broader; the X_i's might for instance be the magnitudes of successive shocks in a sequence of earthquakes or the successive response times of a computer to messages sent via a terminal.

2.1. The first-order autoregressive exponential model (EAR(1))

In a Poisson process the intervals $\{X_i\}$ are independent and identically distributed (i.i.d.) random variables with exponential distribution

$$F_X(x) = 1 - e^{-\lambda x}, \qquad \lambda > 0; \quad x \geq 0 . \tag{2.1}$$

Several attempts have been made to generalize the Poisson process by making the X_i dependent, but with exponential or conditionally exponential marginal distributions (Cox, 1955). The simplest and only really successful attempt in the sense of broad applicability (Gaver and Lewis, 1978) gives a process called the EAR(1) model, derived from the following consideration.

A first-order autoregressive stochastic sequence is defined by the stochastic difference equation (a special case of (1.1))

$$X_i = \rho X_{i-1} + \epsilon_i, \qquad i=0, \pm 1, \pm 2, \ldots; \quad |\rho| < 1 , \tag{2.2}$$

where the ϵ_i are assumed to be an i.i.d. stationary random sequence. If the ϵ_i are normally distributed, so are the X_i. What must the distribution of the ϵ_i be in order for the X_i sequence to be stationary with an exponential(λ) distribution? The answer is surprisingly easy (Gaver and Lewis, 1978).

Let $0 \leq \rho < 1$, and let $\{E_i\}$ be an i.i.d. exponential(λ) sequence. Now let ϵ_i be equal to zero with probability ρ and equal to E_i with probability $1-\rho$. Then we have

$$X_i = \begin{cases} \rho X_{i-1} & \text{probability } \rho \,, \\ \rho X_{i-1} + E_i & \text{probability } (1-\rho) \,, \end{cases} \tag{2.3}$$

$$= \rho X_{i-1} + V_i E_i \,, \tag{2.4}$$

where $\{V_i\}$ is an i.i.d. binary sequence and $P\{V_i=0\} = 1 - P\{V_i=1\} = \rho$. Moreover if we let $X_0 = E_0$, and define X_i as in (2.3), the resulting sequence is stationary for $i=0,1, \ldots$.

The point process with the interval structure (4.3) is called the EAR(1) point process. It is a tractable model, and most of its important properties are given in Gaver and Lewis (1978). In particular we have that $\rho(k) = \rho^k$. This model is in a sense degenerate because it contains runs of X_i in which values are exactly ρ times the previous value; it could, however, be a reasonable model for point processes observed in computer systems (e.g., inter-arrival times of requests to a storage subsystem) in which the intervals have exponential marginal distributions but are dependent. Note that as defined the model can only

provide sequences $\{X_i\}$ with positive serial correlations. We can, however, define the process to include negative correlations (Gaver and Lewis, 1978); there is also a way to obviate the degeneracy (Lawrance, 1978).

Simple generalizations of this first-order, autoregressive, Markovian exponential process are the following.

2.2. The moving average exponential model (EMA(q)).

We define another stationary sequence $\{X_i\}$, using the $\{E_i\}$ sequence above, according to

$$X_0 = E_0 , \tag{2.5}$$

$$X_i = \beta E_i + U_i E_{i-1}, \qquad i=1, \ldots; \quad 0 \leq \beta \leq 1 , \tag{2.6}$$

where $\{U_i\}$ is an i.i.d. binary sequence in which $U_i = 1$ with probability $(1-\beta)$. This is a first order exponential moving average process (EMA(1)) (Lawrance and Lewis, 1977) which is one-dependent; in particular

$$\rho(1) = \beta(1-\beta) \tag{2.7}$$

$$\rho(k) = 0 , \qquad k=2, 3, \ldots . \tag{2.8}$$

Properties of the EMA(1) process are given by Lawrance and Lewis (1977).

It is easy to see that we can make E_{i-1} in (2.6) a random linear combination of E_{i-1} and E_{i-2} to get an EMA(2) process, and can continue the process back q steps to obtain an EMA(q) process. The general EMA(q) model takes the form

$$X_i = \begin{cases} \beta_q E_i & \text{w.p. } b_{q+1} , \\ \beta_q E_i + \beta_{q-1} E_{i-1} & \text{w.p. } b_q , \\ \quad \cdots\cdots & \quad \cdots\cdots \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \ldots + \beta_1 E_{i-q+1} & \text{w.p. } b_2 , \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \ldots + \beta_1 E_{i-q+1} + E_{i-q} & \text{w.p. } b_1 , \end{cases} \tag{2.9}$$

for $0 \leq \beta_1, \beta_2, \ldots, \beta_q \leq 1$; $i=0, \pm 1, \pm 2, \ldots$ and

$$b_i = \begin{cases} \beta_q & i = q+1 , \\ (1-\beta_q) \ldots (1-\beta_i)\beta_{i-1} & q \geq i \geq 2 , \\ (1-\beta_q) \ldots (1-\beta_i) & i = 1 . \end{cases} \tag{2.10}$$

Note that the β_i's can be obtained uniquely from the b_i's; there are $q+1$ b_i's but only q β's, since the sum of the b_i's is equal to one.

This model is clearly only q dependent; in particular the correlations for the EMA(q) process are

$$\rho^{(q)}(k) = \text{corr}(X_i, X_{i-k}) = \begin{cases} \sum_{v=1}^{q-k+1} b_v b_{v+k} & 1 \leq k \leq q , \\ 0 & q+1 \leq k < \infty . \end{cases} \tag{2.11}$$

Thus the serial correlations are just lagged products of the b_i sequence and the formula (2.11) is completely analogous to the formula for the serial correlations of the standard MA(q) process; see Box and Jenkins (1970, p. 68). It can be seem from (2.11) that all the correlations are nonnegative and it may be further shown that they are bounded above by 1/4.

2.3. The EARMA(1,1) model.

By making E_{i-q} in (2.9) autoregressive over the previous E_i's, we obtain a mixed qth order moving-average, first order autoregressive process which we denote by EARMA(1,q). Consider explicitly the case $q=1$. The first order moving-average and first order autoregressive process EARMA(1,1) is given by

$$X_i = \beta E_i + U_i A_{i-1} , \tag{2.12}$$

with

$$A_{i-1} = \rho A_{i-2} + V_i E_{i-1} , \tag{2.13}$$

for $i=1, 2, 3, \ldots$ and $A_{-1} = E_{-1}$ with U_i and V_i as defined above. This sequence of random variables is not Markovian.

The second-order correlation structure of the process is given by

$$\rho(k) = \rho^{k-1} c(\beta,\rho) , \tag{2.14}$$

where

$$c(\beta,\rho) = \beta(1-\beta) + \rho(1-\beta)(1-2\beta) . \tag{2.15}$$

The point process whose intervals have the EARMA(1,1) structure is discussed in detail in Jacobs and Lewis (1977). In particular, for $\beta=1$ it is a Poisson process. The process is very simple to generate on a computer and is very useful for modelling dependent sequences in queueing systems (Jacobs, 1978; Lewis and Shedler, 1978).

2.4. The pth-order autoregressive model EAR(p).

Quite recently ways have been found to obtain exponential sequences $\{X_i\}$ which have autoregressive structure of order p, and to combine these with the moving average process to get a mixed autoregressive-moving

average process EARMA(p,q); see Lewis and Lawrance (1978). Another method of defining pth-order autoregressive exponential sequences, which is closely related to the DARMA(p,q) process discussed later, and which we have only just begun to explore, is described here.

This pth-order exponential autoregressive model can be written as

$$X_i = \alpha_{S_i} X_{i-S_i} + \epsilon_{i,S_i} , \tag{2.16}$$

where the S_i's are i.i.d. discrete random variables taking values 1, 2, ..., p, and ϵ_{i,S_i} is defined to be 0 w.p. α_j, and E_i w.p. α_j if $S_i = j$. If one assume stationarity and that X_{i-1}, X_{i-2}, are marginally exponential(λ), then X_i is a random mixture of E_i and X_{i-1}, ..., X_{i-p} and is exponential(λ). The correlation equations from this process are variants of the familiar Yule-Walker equations. The model is more tractable than the pth-order autoregressive process given in Lewis and Lawrance (1978) and is probably simpler to extend to other distributions than the exponential.

A drawback of these EARMA-type processes is that the serial correlations are all positive, although the scheme given in Gaver and Lewis (1978) for a negatively correlated EARMA1 process can probably be extended to the complete EARMA(p,q) process.

2.5. The semi-Markov generated point process with fixed marginal distribution.

The question arises as to whether there are interval processes $\{X_i\}$ with exponential marginal distributions and, for example, ARMA(1,1) second-order correlation structure and which cover a broader range of correlation than the EARMA(1,1) process (though perhaps at a cost of more complicated structure).

We discuss briefly one such process. It is a special case of the semi-Markov generated point process introduced by Cox (1962) and extended by Haskell and Lewis (1978). We first describe the two-state semi-Markov generated model. In this model there are two types of intervals with distributions $F_1(x)$ and $F_2(x)$, sampled in accordance with a two-state Markov chain for which the one-step transition matrix is

$$\underline{P} = \begin{pmatrix} \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & \alpha_2 \end{pmatrix} \tag{2.18}$$

and the stationary vector is

$$\underline{\Pi} = \underline{\Pi}\ \underline{P} = \left(\frac{1 - \alpha_2}{2 - \alpha_1 - \alpha_2} , \frac{1 - \alpha_1}{2 - \alpha_1 - \alpha_1} \right) . \tag{2.19}$$

When we form the point process we assume that no information is available about the type of interval, i.e., that in the actual bivariate point process of transitions we suppress knowledge of the type of transition. Then the distribution of an interval between transitions (events) X_i in the stationary point process is

$$F_X(x) = \pi_1 F_1(x) + \pi_2 F_2(x) \tag{2.30}$$

and the correlation between X_i and X_{i+k} is

$$\rho(k) = M^{\ k} , \qquad k=1, 2, \ldots , \tag{2.21}$$

where M is a positive constant and $\beta = \alpha_1 + \alpha_2 - 1 = \alpha_1 (1 - \alpha_2)$. Thus the correlation structure is that of an ARMA(1,1) process. For a derivation of this result see Cox and Lewis (1966, Ch. 7, 194-196).

Lewis and Shedler (1973) used this process to model the page exception process in a multiprogrammed computer system. The problem is to deal with the mixture distribution (2.20) for the marginal distribution of intervals; this seams to limit the utility of the model. However, there is a way around it which produces a marginally controlled semi-Markov generated process.

To obtain an exponential marginal distribution, consider the following device (Jacobs and Lewis, 1977). Fix x_0, where $0 < x_0 < \infty$, and let

$$F_1(x) = \begin{cases} \dfrac{\int_0^x e^{-\lambda u} du}{1 - e^{-\lambda x_0}} & 0 \leq x \leq x_0 , \\ 1 & x > x_0 ; \end{cases}$$

$$F_2(x) = \begin{cases} 0 & x \leq x_0 , \\ \dfrac{\int_0^x e^{-\lambda u} du}{e^{-\lambda x_0}} & x > x_0 ; \end{cases} \tag{2.22}$$

then $F(x)$, the marginal distribution of an interval, is exponential(λ) if we set $\pi_1 = 1 - \exp(-\lambda x_0)$. There is one degree of freedom left in the matrix $\underline{P}$; in addition to λ, we have free parameters π_1 (or x_0) and α_1 although the range of α_1 is restricted. What then is the range of β, and can it be negative?

Straightforward manipulation shows that

$$\beta = \frac{\pi_1 - \alpha_1}{\pi_1 - 1} , \tag{2.23}$$

which lies in absolute value between zero and one but can be negative; therefore the serial correlations can be negative. Thus the model appears to be broader than the EARMA(1,1) model. The question of comparing the two models when β is positive has not yet been explored; it requires higher order interval correlations, as discussed by Brillinger (1972).

2.6. Generalizations

The marginally controlled semi-Markov generated sequence $\{X_i\}$ discussed above can be extended in such a way that X_i will have any distribution, say $F(x)$. Thus we let

$$F_1(x) = \begin{cases} \dfrac{F(x)}{F(x_0)} & 0 \le x \le x_0 \,, \\ 1 & x > x_0 \,; \end{cases}$$

$$F_2(x) = \begin{cases} 0 & x \le x_0 \,, \\ \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & x > x_0 \,; \end{cases} \qquad (2.24)$$

then the marginal distribution of an interval is equal to $F(x)$, from (2.30), if we set $\pi_1 = F(x_0)$. Note that the model is very non-linear and the correlation structure is a complicated function of the functional form of $F(x)$.

The much simpler EARMA structure can be extended to some extent. Random variables for which the equation (2.2) has a proper solution are called self-decomposable random variables on random variables of type L. This class includes random variables with Gamma, Cauchy, Pareto, double exponential and perhaps many other distributions. For these random

variables, a pth-order-autoregressive process can be defined as at (2.16). The unique feature of the exponential process is that the ϵ_i which makes X_i exponential(λ) in (2.2) is again an exponential(λ) random variable, albeit mixed with an atom at zero. This property, shared with the double exponential and normal random variables, is what makes it simple to define a moving-average type process, as at (2.9).

3. Count Models: Sequences of discrete-valued random variables.

As remarked earlier, most data on point processes is recorded as numbers of events in successive fixed-length intervals. Despite this fact, most point process models assume that exact times of events are known and it is not simple to derive from these models the statistics of the counts in fixed intervals. Thus in this area in particular flexible models for discrete-valued random variables are needed.

Another application might be to modelling of air pollution data in which concentrations of various chemicals in the air is indicated on a scale of zero to ten. In general this situation requires multivariate time series, but space prohibits discussion of multivariate versions of the DARMA-type processes discussed in this section.

3.1. The first-order autoregressive discrete model (DAR(1)).

Again we denote the sequence of discrete-valued random variables by $\{X_i\}$. If the X_i are counts in a Poisson process then the X_i's are i.i.d. Poisson-distributed random variables. Once dependence is observed in data it is useful to assume, as a first cut, that the dependence is Markovian and use a Markov chain model in which the distribution of X_{i+1} depends only on the value of X_i and is specified by the transition matrix $\underline{P}$ with elements

$$P(k,j) = P\{X_{i+1} = j \mid X_i = k\} , \qquad (3.1)$$

with j and k taking values in the space E, a discrete subset of the real line. Under suitable conditions there is a stationary distribution $\underline{\pi}$ for $\{X_i\}$ given by the equation

$$\underline{\pi} = \underline{\pi}\ \underline{P} . \qquad (3.2)$$

The Markov chain model (3.2) is by virtue of its place in the statistician's toolbox the discrete counterpart of the AR1 process. However the AR1 process has one dependency parameter ρ, plus any parameters which specify the distribution of the ϵ_i's. The Markov chain on the other hand can have an infinite number of parameters and in many cases $\underline{\pi}$ cannot be obtained explicitly from (3.2). This is awkward for statistical analysis. A solution is given by constructing the DAR(1) model (discrete autoregressive model of under one) which is an analog of the EAR(1) model, as follows.

Let Y_i be an i.i.d. sequence of random variables taking values in the space E, and let V_i be an i.i.d. binomial sequence for which $P\{V_i = 1\} = \rho$. Then

$$X_i = V_i X_{i-1} + (1 - V_i)Y_i \qquad i=0, \pm 1, \pm 2, \ldots; \quad 0 \le \rho < 1 . \qquad (3.3)$$

$$= \begin{cases} X_{i-1} & \text{w.p. } \rho , \\ Y_i & \text{w.p. } (1-\rho) . \end{cases} \qquad (3.4)$$

If X_0 has distribution π, then so does X_1 since it is a mixture of two random variables, X_0 and Y_1, with distribution π. Consequently all the X_i, $i=1, 2, \ldots$ have marginal distribution π.

Note that $\{X_i\}$ is a Markov chain with transition probabilities

$$P(k,j) = P\{X_{i+1} = j \mid X_i = k\} = \begin{cases} (1-\rho)\ \pi(j) & k \neq j\ , \\ \rho + (1-\rho)\ \pi(j) & k = j\ ; \end{cases} \qquad (3.5)$$

in fact it is a Markov chain in which the correlation structure is specified by one parameter ρ, and with specified marginal (stationary) distribution $\underline{\pi}$. Thus π may be a Poisson distribution and then the DAR1 model is a 2-parameter (λ,ρ) Markov chain. The analogy with the AR(1) model is clear

As with the EAR(1) model the serial correlations are $\rho(k) = \rho^k \geq 0$. Extensions to negatively correlated sequences are given in Jacobs and Lewis (1978).

3.2. <u>The pth-order autoregressive discrete model (DAR(p))</u>.

First order Markov dependence is a special kind of dependence which is attractive because of analytical tractability considerations, but it is not necessarily met with in practice. One immediate consequence of the Markovian property is that runs of distinct values, say $X_i = j$, have a length which is geometrically distributed (Jacobs and Lewis, 1978a) and this is easily checked in data. If the data fails to have this property, what other types of dependency can be utilized?

A first direction might be to go to higher order (say pth-order) autoregression, which is an explicit pth-order Markov structure, and the DAR(1) model can be extended in this direction. Thus in addition to the assumptions at (3.3) let A_i be an i.i.d. sequence of random variables taking values in $\{1, 2, \ldots, p\}$, with $P\{A_i = j\} = \alpha_j$. Then the DAR(p) process is defined as

$$X_i = V_i X_{i-A_i} + (1 - V_i)Y_i , \qquad i=0, \pm 1, \pm 2, \ldots \tag{3.6}$$

so that X_i is (exclusively) either one of the previous p values X_{i-1}, ..., X_{i-p}, or the error term Y_i. Properties of this model are developed extensively in Jacobs and Lewis (1978c). When $\alpha_1 = 1$, and all other α_j's are zero it is the DAR(1) model.

Yule-Walker equations for the correlations in the stationary DAR(p) process are given in Jacobs and Lewis (1973c) as well as stationarity conditions. In particular for $p=2$ we have the limiting result

$$\nu(k,j) = \lim_{i\to\infty} P\{X_{i+1}=k, X_{i+2}=j\} = \begin{cases} \{1-\rho(1)\}\pi(k)\pi(j) & k \neq j , \\ \rho(1)\pi(j) + \{1-\rho(1)\}\pi(j)^2 & k = j , \end{cases} \tag{3.7}$$

where $\rho(1) = \text{corr}(X_i, X_{i+1})$ in the stationary process. Thus, if we let X_0 and X_{-1} have the joint distribution $\nu(k,j)$, a stationary, second-order autoregressive process with any marginal distribution can be generated. A scheme for obtaining sequences which are possibly negatively correlated is given in Jacobs and Lewis (1978c).

3.3. The q-th order moving average discrete model (DMA(q)).

The other alternative to Markovian dependence (of any order) which is usually considered in time series analysis is the finite-length dependence produced by the moving-average part of the ARMA(p,q) process (1.1). This type of behavior is easily produced for discrete random variables by a random index model of the type

$$X_i = Y_{i-S_i} , \tag{3.8}$$

where S_i are i.i.d. random variables with $P\{S_i \leq k\} = b_k$. Thus we may write

$$X_i = Y_{i-k} \qquad \text{w.p.} \quad b_k - b_{k-1} , \qquad k=0, \ldots, q; \quad b_{-1} = 0 . \tag{3.9}$$

The autoregressive process DAR(p) is also a random index model, but the random indices are not independent. The correlation structure of this DMA(q) process is easily found to be

$$\begin{aligned} \overset{(q)}{\rho}(k) = \operatorname{corr}(X_i X_{i-k}) &= \sum_{v=0}^{q-k} b_v b_{v+k} && 1 \leq k \leq q , \\ &= 0 && k > q . \end{aligned} \tag{3.10}$$

This is the exact analog of (2.11) for the EMA(q) process and the corresponding formula for the MA(q) process. Note that the DMA(q) process is not Markovian. Runs properties of the process are given in Jacobs and Lewis (1978a); the runs are not geometrically distributed.

3.4. <u>Mixed autoregressive-moving average discrete models</u>.

As in the case of the ARMA(p,q) model (1.1), it is useful to have both autoregressive, Markovian dependence and moving average dependence combined into one model. In Jacobs and Lewis (1978a) this was done by replacing the Y_{i-q} term in (3.8) by a discrete autoregression (3.3) over $Y_{i-q}, Y_{i-q-1}, \ldots$. Clearly this can be extended by replacing Y_{i-q} by a p-th order autoregression (3.6) over $Y_{i-q}, Y_{i-q-1}, \ldots$ to obtain a DARMA(p,q) model which is the analog of the EARMA(p,q) model of Lawrance and Lewis (1978). This is not a complete analog of the ARMA(p,q) model in that there is no cross-over of the autoregression and

the moving average, but it is in fact possible to do this to obtain a model called NDARMA(p,q) as follows:

Let

$$X_i = V_i X_{i-A_i} + (1 - V_i) Y_{i-S_i} \qquad i=0, \pm 1, \pm 2, \ldots , \tag{3.11}$$

where the A_i are i.i.d. random variables taking values in $\{1, 2, \ldots, p\}$ with $P\{A_i=j\} = \alpha_j$; the S_i are i.i.d. random variables taking values in $\{0, \ldots, q\}$ with $P\{S_i \leq k\} = F(k)$ and the V_i's are i.i.d. Bernoulli random variables with $P\{V_i=1\} = \rho$.

The model works because a mixture of <u>dependent</u> random variables, all with marginal distribution π, has distribution π; thus if $X_{i-1}, \ldots, X_{i-p}$ have marginal distribution π, then so will X_i since it is a mixture of the dependent random variables $X_{i-1}, \ldots, X_{i-p}$ and $Y_i, \ldots, Y_{i-q}$. Note that when $\rho=0$ we have the DMA(q) process; if in addition $F(0) = 1$ the sequence is i.i.d. since $X_i = Y_i$. When $1 > \rho \neq 0$, $F(0) = 1$ we have the DAR(p) process. Thus the parameters are such that interesting special cases fall out easily. Moreover the ρ parameter measures the degree of mixture of Markovian and moving average dependence, and the distributions of the A_i's and S_i's give a picture of where the dependence is lagged over previous X_i or Y_i values.

The model (3.11) has not yet been fully explored. At first sight it seems preferable to the DARMA(p,q) model, possibly because of the compactness of (3.11) and its close analogy to ARMA(p,q) models. The DARMA(p,q) and NDARMA(p,q) models are, however, distinct and in fact preliminary investigation of the (1.1) case shows that the DARMA(1,1) model (Jacobs and Lewis, 1978a) has a broader correlation structure than does the

NDARMA(1,1). On the other hand the autoregression is not explicit in the DARMA(1,1) model. Both models, therefore, will probably be useful in modelling discrete data such as occur in sampled point processes.

3.5. The marginally controlled semi-Markov generated process.

In the structure of the 2-state marginally controlled semi-Markov generated process detailed at (2.24) no assumption was made about continuity of $F(x)$. Thus $F(x)$ could be discrete, giving a sequence $\{X_i\}$ with known discrete marginal distribution $F(x)$ and ARMA(1,1) correlation structure. By going to an n-state semi-Markov model, a process with ARMA(p,q) correlation structure can be generated (Haskell and Lewis, 1978) with n a function of p and q, and the procedure to obtain a given marginal distribution is just an extension of (2.24). Thus we have, in terms of the quantification of the process by marginal distribution and correlation structure, a direct competitor to the DARMA-type processes.

Comparison of the two types of discrete processes is interesting and points up the simplicity of the DARMA-type processes. In particular the correlation structure of the DARMA(p,q) process is explicit in form if not in detail and the process is a simple, random linear combination of random variables generated from an i.i.d. sequence Y_i. This is clearly not true for the marginally controlled semi-Markov generated process; the recognition that its correlation structure is ARMA-type is accidental and not intuitive. Deeper comparison of these processes in terms, say, of the range of correlation the model will encompass will be instructive. Here again the DARMA-type processes have an advantage; their correlation structure is independent of the marginal distribution π.

4. Summary and Conclusions

We have outlined in this paper three models for discrete-valued and positive-valued time series, all of which to some degree satisfy the criteria of flexibility or simplicity or both set forth in the introduction. Perhaps the main point about the models is that they are designed to accomodate situations in which the marginal distributions in the stationary processes are given and are non-normal.

Properties of these models such as mixing and asymptotic results, higher-order moments, distributions of runs for the discrete models and sums of random variables and point spectra are considered in the references.

There are many other properties of the processes which are still to be explored. Statistical estimation, except in an ad hoc manner and for the Markovian cases, is difficult and has yet to be examined. Extensions to multivariate cases is of great interest for real applications and has been done to some degree in the context of queues with correlated service and arrival times (Jacobs, 1978, and Lewis and Shedler, 1978). The DARMA-type processes, in particular, can be easily extended to coupled equations in the same way as linear processes are extended in econometric models. They might therefore find use in modelling multivariate situations such as the number of cars passing different points in a road evaluated in successive fixed time intervals.

Finally an important problem is to extend the models so as to include inhomogeneity, particularly of the seasonal type, and the effects of concomittant or auxilliary variables. Several schemes are under consideration for these extensions of the models.

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